$x = (x_1 - x_2)/(2l)$, $y = (x_1 + x_2)/(2l)$, we obtain an equation describing the motion in the mode $y = 0, x = x(t)$ (analogous to (1.2)) and an equation governing the orbital stability to a first approximation (analogous to (1.4)), which have the following form

$$x^{\cdot\cdot} + 2a_2x + 4a_4x^3 + 6a_6x^5 = 0$$
$$y^{\cdot\cdot} + 2(e_0 + e_2x^2 + e_4x^4)\, y = 0$$
$$a_2 = \tau_1/2 + \tau_2, \quad a_4 = \gamma_1/8 + \gamma_2\sigma, \quad a_6 = -\gamma_1/16 - 2\gamma_2\sigma^2$$
$$e_0 = \tau_1/2, \quad e_2 = 3\gamma_1/4, \quad e_4 = -15\gamma_1/16$$

In this case conditions (1.9) result in the relationships

$$3 = N(1 + 8\gamma_3), \quad 15 = 4N(1 + 32\gamma_3\sigma)$$
$$8(\tau_1 + 2\tau_2)(1 + 8\gamma_3)(1 + 32\gamma_3\sigma) + (1 + 8\gamma_3)^3 = 4h(1 + 32\gamma_3\sigma)^2$$
$$\gamma_3 = \gamma_2\sigma/\gamma_1, \quad N = n(n + 1)$$

Here n is the number of bounded instability zones $n = 0, 1, 2, \ldots$

For instance, let $n = 1$. Then there should be $\sigma = 7/16, \gamma_3 = 1/16, 40(\tau_1 + 2\tau_2) + 6 = 25h$.

Near the values of the system parameters and the energy h that satisfy the above relationships, all except n instability zones are "narrow" since they shrink to a line in the case when the relationships are satisfied exactly. This also refers to the first parametric resonance zone, which is ordinarily assumed to be "wide".

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ON THE EXTREMALITY HYPOTHESIS OF STABLE RESONANCE MOTIONS*

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On the basis of a proposed approximate method of determining the mean values of functions of the coordinates and time on almost-integrable trajectories of dynamic systems, the force function and kinetic energy are averaged in the following problems: the motion of a material point in the neighbourhood of triangular points of libration of the plane circular restricted three-body problem, the motion of a physical pendulum with a rapidly oscillating point of suspension in the neighbourhoods of the lower and upper equilibrium positions. Preference is shown for the following hypotheses: the minimum of the averaged potential (V.V. Beletskii hypothesis), kinetic, and total energy of the mechanical system at stable, isolated, synchronous motions.

The extremal principle proposed in the form of the V.V. Beletskii hypothesis /2, 3/ is of special interest in investigations of the extremal properties of stable resonance (synchronous) motions: The function

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$$\langle U\rangle = f(x_0) = \lim_{t\to\infty} \frac{1}{t}\int_0^t U(x(x_0,t),t)\,dt$$

takes maximum values in stable resonance motions. Here U is a force function depending periodically on the time t, while $x(x_0, t)$ is the solution of the equations of motion of a mechanical systems with initial value x_0 at the time $t = 0$. In connection with this extremal principle, the extremal properties of stable synchronized motions were studied intensively (/4–6/, for example) without, however, removing the question of its validity.

Analysis of the values of the function $\langle U\rangle$ is difficult in principle, because the law of motion $x(x_0, t)$ is represented by a non-integrable differential system of equations of motion. The approximate evaluation of $\langle U\rangle$ on a computer, for the problem of plane satellite rotations relative to its centre of mass moving in an elliptic orbit. * (See /2/, as well as Beletskii, V.V. and Shliakhtin, A.N., Resonance rotations of a satellite when the magnetic and gravitational fields interact. Preprint of the Institute of Applied Mathematics of the USSR Academy of Sciences, No.46, 1980.) yielded results in agreement with the hypothesis.

Such an approximate approach to the determination of $\langle U\rangle$ as well as the mean, with respect to time, of some other functions can be realized by analytical methods in certain problems. We consider the class of mechanical systems containing a small parameter ε and which are integrable for $\varepsilon = 0$. The following scheme for the approximate evaluation of $\langle U\rangle$ can be proposed for it:

1) We apply the method of averaging or the method of the small parameter to the system of equations of motion since they enable solutions to be found close to the exact solutions in fairly long time intervals;

2) We formally replace the exact (unknown) solutions by the approximate solutions obtained, by means of which we also take the average of the force function U.

It is understood that in such an approach it is impossible to guarantee the rigour of the results obtained, but its application results in interesting hypothetical deductions.

1. We consider the plane restricted circular three-body problem /7/. Let μ and $1-\mu$ be the masses of two material points separated by unit distance. These points perform circular motion at unit angular velocity relative to their common centre of mass O. We take an Oxy coordinate system rotating at unit angular velocity such that μ and $1-\mu$ are at rest on the Ox axis. Let a weightless material point $P(x, y)$ move in a gravitational field of mass μ and $1-\mu$. Its kinetic energy has the form

$$T = {}^1/_2\,[x^{\cdot 2} + y^{\cdot 2} + 2(xy^\cdot - x^\cdot y) + x^2 + y^2], \quad x^\cdot = \frac{dx}{dt}, \quad y^\cdot = \frac{dy}{dt}$$

The force function of attraction of the point P is given by the formula

$$U(x, y) = (1-\mu)\{(x+\mu)^2 + y^2\}^{-1/2} + \mu\{[x-(1-\mu)]^2 + y^2\}^{-1/2}$$

Among the set of motions of P there is a stable equilibrium position /7–9/ in the variables $x, y, x^\cdot, y^\cdot$ for $\mu(1-\mu) < 1/27$, $\mu \neq (15-\sqrt{213})/30$, $\mu \neq (45-\sqrt{1833})/90$, namely, the triangular libration point $L_4(x_\Delta, y_\Delta)$: $x_\Delta = {}^1/_2 - \mu$; $y_\Delta = \sqrt{3}/2$.

We perform a transformation from x, y to their perturbations x_1, x_3 with respect to L_4: $x = ({}^1/_2 - \mu) + x_1$; $y = \sqrt{3}/2 + x_3$, then neglecting negligible constants, T and U can be represented in the form of power series in the variables x_1, x_3 and their derivatives with respect to time $x_1^\cdot, x_3^\cdot$.

The motion of the point P is characterized by the following differential equations:

$$x_1^\cdot = x_2, \quad x_2^\cdot = \frac{3}{4}x_1 + \frac{3\sqrt{3}}{4}(1-2\mu)x_3 + 2x_4 + \frac{21}{16}(1-2\mu)x_1^2 - \frac{3\sqrt{3}}{8}x_1x_3 - \frac{33}{16}(1-2\mu)x_3^2 + \ldots \tag{1.1}$$

$$x_3^\cdot = x_4, \quad x_4^\cdot = \frac{3\sqrt{3}}{4}(1-2\mu)x_1 - 2x_2 + \frac{9}{4}x_3 - \frac{3\sqrt{3}}{16}x_1^2 - \frac{33}{8}(1-2\mu)x_1x_3 - \frac{9\sqrt{3}}{16}x_3^2 + \ldots$$

If only first-order terms are kept, we then obtain equations in variations with respect to L_4 whose general solution is represented for $\mu(1-\mu) < 1/27$ in the form

$$x_{1(0)} = A_1z_1 + B_1z_2 + A_2z_3 + B_2z_4, \quad x_{2(0)} = x_{1(0)}^\cdot$$
$$x_{3(0)} = C_1z_1 + D_1z_2 + C_2z_3 + D_2z_4, \quad x_{4(0)} = x_{3(0)}^\cdot$$
$$\Big(A_n = \frac{3\sqrt{3}}{4}(1-2\mu)\sin\omega_n t + 2\omega_n\cos\omega_n t, \quad B_n = -\Big(\frac{9}{4} + \omega_n^2\Big)\cos\omega_n t,$$
$$C_n = -\Big(\frac{3}{4} + \omega_n^2\Big)\sin\omega_n t, \quad D_n = \frac{3\sqrt{3}}{4}(1-2\mu)\cos\omega_n t + 2\omega_n\sin\omega_n t, \ n = 1, 2\Big)$$

In these formulas ω_1, ω_2 are the eigenfrequencies determined by the characteristic equation

$$\Big(\omega^2 + \frac{3}{4}\Big)\Big(\omega^2 + \frac{9}{4}\Big) - 4\omega^2 - \frac{27}{16}(1-2\mu)^2 = 0 \tag{1.2}$$

and z_i are arbitrary constants related by a linear transformation to the initial values x_i° of the variables x_i.

By Poincaré's small parameter method, the solution of the complete system (1.1) can be represented by power series in z_i, or equivalently, in x_i°

$$x_i = \sum_{j=1}^{4} A_{ij}(t)\, z_j + \sum_{j,\,k=1}^{4} B_{ijk}(t)\, z_j z_k + \ldots, \qquad x_i^\circ = \sum_{j=1}^{4} A_{ij}(0)\, z_j,\ B_{ijk}(0) = 0, \ldots \tag{1.3}$$

For a sufficiently small norm of the vector $z \in R^4$, these series reduce to a fairly large, but finite, time interval $[0, \tau]$, where $\tau \to \infty$ as $\|z\| \to 0$.

We now take the average of the force function U in the set of trajectories given by (1.3) for $t \in [0, \infty)$. To the accuracy of second-order quantities in the coordinates of the vector z_i inclusive, we obtain

$$\langle U\rangle = -\frac{3}{4}\mu(1-\mu)\left[z_1^2\left(\frac{3}{4}+\omega_1^2\right) + 2z_1z_2(-2\omega_1) + z_2^2\left(\frac{9}{4}+\omega_1^2\right) + z_3^2\left(\frac{3}{4}+\omega_2^2\right) + 2z_3z^4(-2\omega_2) + z_4^2\left(\frac{9}{4}+\omega_2^2\right)\right] + O(\|z\|^3)$$

Taking (1.2) into account, the positive-definiteness of the quadratic form in z_i in the square brackets can be proved. Therefore, $\langle U\rangle$ as a function of the initial values x_i° takes its maximum value at the point $x_i^\circ = 0$, i.e., at the triangular libration point L_4 (L_4 is stable isolated synchronized motion).

The same averaging of the Lagrange function $T + U$ shows that to second powers in z_i inclusive, $\langle T + U\rangle = 0$. This means that $\langle T\rangle$ takes a minimum value at L_4.

Analogous deductions hold for the other triangular libration point L_5.

Remark. After replacing z_i by x_i° the function $\langle U\rangle$ acquires the denominators ω_1, $\omega_2^2 - \omega_1^2$, and consequently becomes undetermined if

1) $\omega_1 = 0$, i.e. $\mu = 0$. The limit case of the problem under consideration, which represents the Kepler problem, corresponds to this value of μ. The motion corresponding to the limit position of L_4 is unstable;

2) $\omega_2 = \omega_1$. We are at the boundary point of the domain for satisfying the necessary conditions for stability: $\mu(1-\mu) = 1/27$.

We hence deduce that $\langle U\rangle$ reacts to critical values of μ at which the property of stability of the triangular libration points is lost.

2. We examine the problem of the motion of a physical pendulum which can rotate in a definite vertical plane around its point of suspension that performs small-amplitude and high-frequency sinusoidal oscillations in the vertical direction. The set of pendulum motions contains two equilibrium positions when the pendulum is directed upward and downward along the vertical. Let θ be the angle of declination from the lower equilibrium position; then the motion will be described by the following differential equation (/10/,p.371):

$$\theta^{\cdot\cdot} + (k^2\varepsilon^2 - \varepsilon\sin t)\sin\theta,\ \ k^2 < 1,\ \ 0 < \varepsilon \ll 1 \tag{2.1}$$

We take the expressions

$$T = {}^1/_2\theta^{\cdot 2}, \qquad U = -2(k^2\varepsilon^2 - \varepsilon\sin t)\sin^2\frac{\theta}{2}$$

as the kinetic energy and the force function.

We consider pendulum motion near the lower equilibrium position $\theta(t) = 0$ ($\theta(t) = 0$ is stable isolated synchronous motion). We use the substitution /10/

$$\theta = x - \varepsilon\sin t\sin x,\quad \theta^{\cdot} = \varepsilon y - \varepsilon\cos t\sin x\ \ (\theta, \theta^{\cdot} \to x, y)$$

that reduces (2.1) to a system of standard form

$$x^{\cdot} = \varepsilon y + \varepsilon^2 \ldots,\ \ y^{\cdot} = \varepsilon\,[y\cos t - k^2\sin x - {}^1/_2\sin^2 t\sin 2x] + \varepsilon^2 \ldots \tag{2.2}$$

We take the average of the system (2.2)

$$x = \xi,\ \ y = \eta + \varepsilon\,(\eta\sin t\cos\xi + {}^1/_8\sin 2t\sin 2\xi)$$
$$\xi^{\cdot} = \varepsilon\eta,\ \ \eta^{\cdot} = -\varepsilon\,[k^2\sin\xi + {}^1/_4\sin 2\xi]$$

These equations enable us to find approximate solutions in the neighbourhood of the lower equilibrium position that are close to the exact solutions of (2.2) in a large time interval. Taking the average of U and T with respect to such solutions we obtain

$$\langle U\rangle = -\frac{\varepsilon^2}{4}(1+k^2)\left[\xi_0^2 + \frac{\eta_0^2}{0{,}5+k^2}\right], \qquad \langle T\rangle = -\langle U\rangle$$

to the accuracy of $\varepsilon^2 O(\|\gamma\|^3) + \varepsilon^3 O(\|\gamma\|^2)$, where $\gamma = \{\xi_0, \eta_0\}$ is the vector of the initial values of the variables ξ, η.

We obtain the maximum of $\langle U\rangle$, the minimum of $\langle T\rangle$ for $\xi_0 = \eta_0 = 0$, i.e., even in this case we have the maximum of $\langle U\rangle$, and the minimum of $\langle T\rangle$ at a stable, isolated, synchronous motion, the lower equilibrium position.

We now consider the upper equilibrium positon of the pendulum, which becomes stable for $k^2 < 0{,}5$ /10/. Taking the average of the force function and the kinetic energy just as we did for the lower equilibrium position, we again obtain the maximum of the mean U and the minimum of the mean T at the stable upper equilibrium position.

Therefore, an approximate analysis of the values of $\langle U\rangle, \langle T\rangle$ in specific mechanics problems verifies the V.V. Beletskii hypothesis and also enables us to propose a hypothesis about the minimality of the mean value of the kinetic energy and the minimality of the mean value of the total energy $T - U$ of mechanical systems in stable isolated synchronous motions.

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ON OSCILLATIONS OF A GYROSTAT AROUND STABLE PERMANENT ROTATIONS*

M.P. TSOPA

The oscillations of a gyrostat with constant gyrostatic moment are investigated by the method of averaging for the Euler, Lagrange and Kovalevskaya cases, which are analogous to the oscillations studied earlier /1/ of a solid around its stable permanent rotations.

We consider the perturbed motion of a gyrostat in the neighbourhood of permanent rotations in a central Newtonian field /2/ whose force function is given by

$$U = -mg\,(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) - {}^1/_2\mu\,(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2),\ \ \mu = 3g/R.$$

Here A, B, C are the principal moments of gyrostat inertia, γ_i are the direction cosines of the z-coordinate axis in the principal axes of inertia, x_0, y_0, z_0 are coordinates of the centre of gyrostat mass in the axes of inertia, m is the gyrostat mass, and g is the acceleration due to gravity at a distance R from the gravitating centre.

In the Euler, Lagrange, and Kovalevskaya cases the characteristic equation of the first approximation has one or two zero roots and one or two pairs of pure imaginary roots. Consequently, the transformation from the variables x_i to the amplitudes a_k, ξ_k and phases u_k in matrix form will be

$$x = \sum a_i\,[\operatorname{Re} V_1(i\omega_i)\cos u_i - \operatorname{Im} V_1(i\omega_i)\sin u_i] + V_2(0)\,\xi_i \qquad (1)$$

where V_1, V_2 are non-zero columns for the pure imaginary and zero roots of the associated matrix

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